Selection Rules and the Decomposition of the Kronecker Square of Irreducible Representations*

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The irreducible representations occurring in the decomposition of the Kronecker squares of irreducible representations of finite and continuous groups are shown to be readily separable into symmetric and antisymmetric parts using Littlewood's method of plethysm. Particular applications of the rotation and symplectic groups, together with selection rules for isoscalar factors, are given.

I. INTRODUCTION

THE Kronecker square $\Gamma \times \Gamma$ of an irreducible representation Γ of a group G is always reducible to the sum of a symmetric product representation $[\Gamma^2]$ and an antisymmetric product representation $\{\Gamma^2\}$, such that[1,2]

$$\Gamma \times \Gamma = [\Gamma^2] + \{\Gamma^2\}. \tag{1}$$

The resolution of Kronecker squares into symmetric and antisymmetric product representations is of particular significance in determining selection rules over and above those normally found by the decomposition of the appropriate triple Kronecker products.[1–3]

Judd and Wadzinski[3] have recently discussed the resolution of the Kronecker squares of the irreducible representations of the continuous groups R_7 and G_2. Their method, which basically makes use of a chain calculation starting with the trivial resolution of a few simple representations, rapidly becomes excessively tedious. The procedure of resolving Kronecker squares may be greatly simplified by using the more direct method of Littlewood's[4–9] operation of plethysm. With seemingly only two exceptions,[10,11] the powerful technique of plethysm has been unrecognized by physicists. In the present paper, we briefly review Littlewood's method and then consider its application to our central problem of resolving the Kronecker squares of irreducible representations into their symmetric and antisymmetric parts for both finite and continuous groups.

II. PLETHYSM AND THE GENERAL LINEAR GROUP $GL(n)$

The construction of basis functions $\phi_\alpha^{(\Gamma)}$ that transform according to an irreducible representation Γ of a group G is a common problem in group theory. If the irreducible representation Γ is of degree n, then the basis function $\phi_\alpha^{(\Gamma)}$ also spans the representation $\{1\}$ of the general linear group $GL(n)$. The products $[\phi_\alpha^{(\Gamma)}]^r$ of the basis functions $\phi_\alpha^{(\Gamma)}$ form bases for the different irreducible representations $\{\lambda\}$ of $GL(n)$ contained in the Kronecker product

$$\{1\}^r = \sum g_\lambda\{\lambda\}, \tag{2}$$

where the λ's are partitions of the integer r, and g_λ is the number of times a given irreducible representation $\{\lambda\}$ occurs in the decomposition of the product. The irreducible representations $\{\lambda\}$ of $GL(n)$ generally will be reducible under restriction to the group G.

The product functions constructed from the $\phi_\alpha^{(\Gamma)}$'s that form a basis for the $\{\lambda\}$ representation of $GL(n)$ have the symmetry associated with the corresponding representation (λ) of the symmetric group of order $r!$. In practice we are usually interested in constructing product functions of a particular symmetry type, and hence, in picking out those product functions forming a basis for the representation $\{\nu\}$ of $GL(n)$ which turns up in the sum $\sum g_\lambda\{\lambda\}$ of Eq. (2). The decomposition of the representation $\{\lambda\}$ of $GL(n)$, on restriction to G, can then be studied.

Let us suppose that Γ corresponds to the irreducible representation $\{\mu\}$ of $GL(m)$ and that the functions $\phi_\alpha^{\{\mu\}}$ form a basis for this representation. If the dimension of $\{\mu\}$ is n ($m \leq n$ for $\{\mu\} \neq \{0\}$), then this set of functions also forms a basis for the representation $\{1\}$ of $GL(n)$. We now construct powers and

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[1] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison–Wesley Publishing Company, Inc., Reading, Mass., 1962).
[2] J. S. Griffith, *The Theory of Transition—Metal Ions* (Cambridge University Press, New York, 1961).
[3] B. R. Judd and H. T. Wadzinski, J. Math. Phys. **8**, 2125 (1967).
[4] D. E. Littlewood, J. Lond. Math. Soc. **11**, 49 (1936).
[5] D. E. Littlewood, Phil. Trans. Roy. Soc. London **A239**, 305 (1944).
[6] D. E. Littlewood, Phil. Trans. Roy. Soc. London **A239**, 387 (1944).
[7] D. E. Littlewood, *Theory of Group Characters and Matrix Representations of Groups* (Oxford University Press, New York 1958), 2nd ed.
[8] D. E. Littlewood, *A University Algebra* (William Heinemann Ltd., London, 1950).
[9] D. E. Littlewood, *The Skeleton Key of Mathematics* (Harper and Row, New York, 1960).
[10] M. Kretzschmar, Z. Physik **158**, 284 (1960).
[11] J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 (1958).

products of the functions $\phi_\alpha^{\{\mu\}}$ of degree r, and from these product functions we choose a basis for the representation $\{\nu\}$ of $GL(n)$, where (ν) is a partition of r. In general, these product functions form a basis for a reducible representation of $GL(m)$. If we denote the set of functions forming a basis for the $\{\lambda\}$ representation of $GL(m)$ by $\phi^{\{\lambda\}}$, and if we denote the set of powers and products of the functions $\phi^{\{\mu\}}$ forming a basis for the $\{\nu\}$ representation of $GL(n)$ by $|\phi^{\{\mu\}}|^{\{\nu\}}$, then clearly

$$|\phi^{\{\mu\}}|^{\{\nu\}} = \sum \phi^{\{\lambda\}}. \tag{3}$$

The above result can be expressed equivalently in terms of transformation matrices. The functions $\phi^{\{\mu\}}$ may be expressed as linear combinations of powers and products of functions $\phi^{\{1\}}$ that form a basis for the $\{1\}$ representation of $GL(m)$. The matrix $A^{\{\mu\}}$, which transforms the $\phi^{\{\mu\}}$'s, is then said to be an induced matrix of the matrix A which transforms the functions $\phi^{\{1\}}$. The operation $|\phi^{\{\mu\}}|^{\{\nu\}}$, which we perform on the functions $\phi^{\{\mu\}}$, is equivalent to forming the induced matrix $|A^{\{\mu\}}|^{\{\nu\}}$ of the induced matrix $A^{\{\mu\}}$. An induced matrix of an induced matrix is, in general, reducible to the direct sum of other induced matrices $A^{\{\lambda\}}$, each of which is the transformation matrix appropriate to the irreducible representation $\{\lambda\}$ with basis functions $\phi^{\{\lambda\}}$. Equation (3) is thus equivalent to the equation

$$|A^{\{\mu\}}|^{\{\nu\}} = \dot{\sum} A^{\{\lambda\}} \tag{4}$$

written in terms of the transformation matrices with $\dot{\sum}$ denoting a direct sum.

In Littlewood's development of the character theory of continuous groups, the symbol $\{\lambda\}$ is used to represent an S function which is defined as the spur[9] of the induced matrix $A^{\{\lambda\}}$. The operations physicists usually associate with representations of continuous groups reflect the properties of the S functions corresponding to the various matrices transforming the elements of the representations. For example, the decomposition of a product representation $\{\lambda\} \times \{\mu\}$ into its simple components is equivalent to the expression of the product of two S functions as the sum of S functions. Equation (4) may be taken as defining a particular type of multiplication of S functions.[4] Taking the spurs of the matrices, we have the definitive equation for the plethysm of S functions:

$$\{\mu\} \otimes \{\nu\} = \sum \{\eta\}, \tag{5}$$

where the symbol $\otimes$ is used to indicate the operation of plethysm[12] and $\{\mu\} \otimes \{\nu\}$ is read as "$\{\mu\}$ plethys $\{\nu\}$."

[12] The symbol $\otimes$ is frequently used to designate the Kronecker outer product; here we reserve it solely for the operation of plethysm. In general, we shall follow Littlewood's notation (Ref. 7) throughout.

The basic algebra of plethysm has been developed by Littlewood, and here we simply state the principal results. Plethysm is distributive on the right with respect to both multiplication and addition, i.e.,

$$A \otimes (B + C) = A \otimes B + A \otimes C, \tag{6}$$

and

$$A \otimes (BC) = (A \otimes B)(A \otimes C) = A \otimes BA \otimes C. \tag{7}$$

For addition, subtraction, and multiplication to the left we have:

$$(A + B) \otimes \{\lambda\} = \sum \Gamma_{\nu\mu\lambda}(A \otimes \{\mu\})(B \otimes \{\nu\}), \tag{8}$$

where $\Gamma_{\mu\nu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\mu\}\{\nu\}$;

$$(A - B) \otimes \{\lambda\} = \sum (-1)^r \Gamma_{\mu\nu\lambda}(A \otimes \{\mu\})(B \otimes \{\tilde{\nu}\}), \tag{9}$$

where $\{\tilde{\nu}\}$ is the partition of r conjugate to $\{\nu\}$;

$$(AB) \otimes \{\lambda\} = \sum g_{\mu\nu\lambda}(A \otimes \{\mu\})(B \otimes \{\nu\}), \tag{10}$$

where $g_{\mu\nu\lambda}$ is the coefficient of the character $\chi^{(\lambda)}$ of the symmetric group on n symbols, where $\chi^{(\mu)}\chi^{(\nu)} = \sum g_{\mu\nu\lambda}\chi^{(\lambda)}$, and (μ), (ν), and (λ) are all partitions of n; and

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{11}$$

These formulas may all be readily extended by repeated application of the basic formulas. For example, we readily obtain

$$\begin{aligned}(A + B - C) \otimes \{\lambda\} &= \sum (-1)^r \Gamma_{\mu\nu\lambda}((A + B) \otimes \{\mu\})(C \otimes \{\tilde{\nu}\}),\\ &= \sum (-1)^r \Gamma_{\mu\nu\lambda}\Gamma_{\eta\tau\mu}(A \otimes \{\eta\})(B \otimes \{\tau\})(C \otimes \{\tilde{\nu}\}),\end{aligned} \tag{12}$$

where (ν) is a partition of r. It is also useful to note that, if (λ) is a partition of r and

$$\{\lambda\} \otimes \{\mu\} = \sum \{\nu\},$$

then, if r is *even*,

$$\{\tilde{\lambda}\} \otimes \{\mu\} = \sum \{\tilde{\nu}\}, \tag{13}$$

and, if r is *odd*,

$$\{\tilde{\lambda}\} \otimes \{\tilde{\mu}\} = \sum \{\tilde{\nu}\}, \tag{14}$$

where the sign $\tilde{}$ denotes that the conjugate partition is taken.

The following theorem is particularly useful in the construction of a plethysm: If

$$\{\lambda\} \otimes \{\pi\} = \sum \{\nu\}, \tag{15a}$$

then

$$\sum_{\zeta,\nu} \Gamma_{1\zeta\nu}\{\zeta\} = \left[\sum_{\mu} \Gamma_{1\gamma\pi}\{\lambda\} \otimes \{\gamma\}\right]\left[\sum_{\mu} \Gamma_{1\mu\lambda}\{\mu\}\right]. \tag{15b}$$

If $\{\pi\} \equiv \{n\}$ has only one part, then Eq. (15b) reduces to

$$\sum_{\zeta,\nu} \Gamma_{1\zeta\nu}\{\zeta\} = \{\lambda\} \otimes \{n - 1\}\left[\sum_{\mu} \Gamma_{1\mu\lambda}\{\mu\}\right]. \tag{15c}$$

This last equation forms the basis of Littlewood's "third method."[5] The right-hand sides of Eqs. (15b) and (15c) are usually readily expandable; one is left only with the task of selecting a suitable set of $\{\nu\}$'s to match the left-hand side.

A theorem due to Ibrahim[13] is particularly useful in defining the choice of $\{\nu\}$ in Littlewood's third method. The *principal part* of a product of two S functions $\{\alpha\}$ and $\{\beta\}$ is defined as $\{\alpha_1+\beta_1,\ \alpha_2+\beta_2,\ \alpha_3+\beta_3, + \cdots,\}$. Ibrahim then proves the following: The principal part of the products of terms in the expansion $(\{\lambda\}\otimes\{\omega\})(\{\mu\}\otimes\{\eta\})$, appear as terms in the expansion of

$$\{\lambda_1+\mu_1, \lambda_2+\mu_2, \lambda_3+\mu_3, \cdots,\}\otimes\{\nu\},$$

wherever $\chi^{(\omega)}\chi^{(\eta)}=\chi^{(\nu)}$, where (ω), (η), and (ν) are all partitions of n, and the χ's are characters of S_{nl}. Three special cases found by Ibrahim[14] are of use here.

1. The principal parts in the product

$$(\{\lambda\}\otimes\{n\})(\{\mu\}\otimes\{1^n\})$$

are the terms in the expansion of $\{\lambda_1+\mu_1,\ \lambda_2+\mu_2, \cdots,\}\otimes\{1^n\}$, where (n) is a partition into one part.

2. The principal parts in the product

$$(\{\lambda\}\otimes\{n\})(\{\mu\}\otimes\{n\})$$

are the terms in the expansion of $\{\lambda_1+\mu_1,\ \lambda_2+\mu_2, \cdots,\}\otimes\{n\}$.

3. The principal parts in the product

$$(\{\lambda\}\otimes\{1^n\})(\{\mu\}\otimes\{1^n\})$$

are the terms in the expansion of $\{\lambda_1+\mu_1,\ \lambda_2+\mu_2, \cdots,\}\otimes\{n\}$.

These three results provide a list of S functions which certainly belong to the reduction of the plethysm. While this list is not necessarily complete, it usually permits an unambiguous selection of the $\{\nu\}$'s of Eq. (15a) to be made.

Littlewood[5] has given two results that are of assistance in establishing further plethysms. If n is an integer, then

$$\{n\}\otimes\{2\}=\{2n\}+\{2n-2,2\}+\{2n-4,4\}+\cdots \tag{16}$$

to $(n+1)/2$ or $(n+2)/2$ terms, and

$$\{n\}\otimes\{1^2\}=\{2n-1,1\}+\{2n-3,3\}+\cdots \tag{17}$$

to $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$ terms. Using these two results, together with the conjugation theorems of Eqs. (12) and (13), it is readily deduced for *n odd* that

$$\{1^n\}\otimes\{2\}=\{2^1,1^{2n-2}\}+\{2^3,1^{2n-6}\}+\cdots \tag{18}$$

and

$$\{1^n\}\otimes\{1^2\}=\{1^{2n}\}+\{2^2,1^{2n-4}\} + \{2^4,1^{2n-8}\}+\cdots, \tag{19}$$

both to $\frac{1}{2}(n+1)$ terms, while for *n even*

$$\{1^n\}\otimes\{2\}=\{1^{2n}\}+\{2^2,2^{2n-4}\} + \{2^4,1^{2n-8}\}+\cdots \tag{20}$$

to $\frac{1}{2}(n+2)$ terms, and

$$\{1^n\}\otimes\{1^2\}=\{2^1,1^{2n-2}\}+\{2^3,1^{2n-6}\} + \{2^5,1^{2n-10}\}+\cdots \tag{21}$$

to $\frac{1}{2}n$ terms.

Our problem of resolving the Kronecker square of a representation Γ into its symmetric and antisymmetric product representations amounts to forming the plethysms

$$[\Gamma^2]=\Gamma\otimes\{2\} \tag{22}$$

for the symmetric representations and

$$\{\Gamma^2\}=\Gamma\otimes\{1^2\} \tag{23}$$

for the antisymmetric representation.

Equations (16)–(21) allow many of the Kronecker squares of irreducible representations of the general linear group to be decomposed into their symmetric and antisymmetric representations immediately. To extend these results, it is necessary to make use of Eq. (15b), which reduces to

$$\sum_{\zeta,\nu}\Gamma_{1\zeta\nu}\{\zeta\}=\{\lambda\}\sum_{\mu}\Gamma_{1\mu\lambda}\{\mu\}, \tag{24}$$

if $\{\pi\}\equiv\{2\}$ or $\{1^2\}$ in Eq. (15a). There are two distinct choices for the series of S functions $\{\nu\}$ both of which yield the right-hand side. Ibrahim's three results give the distinction between the two sets.

As an example, consider the case

$$\{21\}\otimes\{2\}=\sum\{\nu\}.$$

Equation (15) gives

$$\begin{aligned}\sum\Gamma_{1\zeta\nu}\{\zeta\}&=\{21\}[\sum\Gamma_{1\mu(21)}\{\mu\}]\\&=\{21\}[\{1^2\}+\{2\}]\\&=\{41\}+2\{32\}+2\{31^2\}+2\{2^21\}+\{21^3\}.\end{aligned}$$

Applying Ibrahim's theorem with $\{\lambda\}\equiv\{1\}$ and $\{\mu\}\equiv\{1^2\}$, we find that $(\{1\}\otimes\{2\})(\{1^2\}\otimes\{2\})$ has principal parts $\{42\}$ and $\{31^3\}$, allowing us to establish the complete list of $\{\nu\}$'s by noting that

$$\begin{aligned}\{42\}&\to\{32\}+\{41\},\\\{31^3\}&\to\{31^2\}+\{21^3\},\\\{2^3\}&\to\{2^21\},\\\{321\}&\to\{2^21\}+\{31^2\}+\{32\}\end{aligned}$$

[13] E. M. Ibrahim, Oxford Quart. J. Math. 3, 50 (1952).
[14] E. M. Ibrahim, Am. Math. Soc. Proc. 7, 199 (1956).

TABLE I. Decomposition of the Kronecker squares of irreducible representations of the general linear group.

$\{\lambda\}$	$\{\lambda\}\otimes\{2\}$	$\{\lambda\}\otimes\{1^2\}$
$\{21\}$	$\{42\}+\{31^3\}+\{321\}+\{2^3\}$	$\{41^2\}+\{3^2\}+\{321\}+\{2^21^2\}$
$\{21^2\}$	$\{42^2\}+\{41^4\}+\{3^21^2\}+\{321^4\}+\{2^4\}+\{2^21^4\}$	$\{421^2\}+\{3^22\}+\{321^3\}+\{31^5\}+\{2^31^2\}$
$\{2^2\}$	$\{4^2\}+\{42^2\}+\{2^21^2\}+\{2^4\}$	$\{431\}+\{32^21\}$
$\{2^21\}$	$\{4^2\}+\{4321\}+\{431^3\}+\{42^3\}+\{3^31\}+\{32^31\}+\{32^21^3\}+\{3^221^2\}+\{2^5\}$	$\{4^21^2\}+\{43^2\}+\{4321\}+\{42^21^2\}+\{3^22^2\}+\{3^221^2\}+\{3^21^4\}+\{32^31\}+\{2^41^2\}$
$\{2^3\}$	$\{4^3\}+\{4^22^2\}+\{43^21^2\}+\{42^4\}+\{3^22^21^2\}+\{2^6\}$	$\{4^231\}+\{432^21\}+\{3^31^3\}+\{32^41\}$

to give

$$\{21\}\otimes\{2\}=\{42\}+\{31^3\}+\{2^3\}+\{321\}.$$

Using Eq. (14) and the result for $\{21\}\otimes\{2\}$ immediately gives

$$\{21\}\otimes\{1^2\}=\{41^2\}+\{3^2\}+\{321\}+\{2^21^2\}.$$

This result may be checked by noting that

$$\{\lambda\}\otimes(\{2\}+\{1^2\})=\{\lambda\}\{\lambda\}. \qquad (25)$$

The method just outlined allows the rapid decomposition of the Kronecker squares of the irreducible representations of the general linear group given in Table I. If $\{\lambda\}$ is an irreducible representation of $GL(m)$, then all partitions occurring in the decomposition that have more than m parts will be null. Individual decompositions may be checked by noting that if the irreducible representation $\{\lambda\}$ of $GL(m)$ is of degree n, then the sum of the dimensions of the symmetric representations must be equal to that of the representation $\{2\}$ of $GL(n)$, i.e., $n[\frac{1}{2}(n+1)]$ while the sum of the antisymmetric representations must be equal to that of the representation $\{1^2\}$ of $GL(n)$, i.e., $n[\frac{1}{2}(n-1)]$. Since the representations of $GL(m)$ remain irreducible under restriction to the unitary group $U(m)$, the entries in Table I may be equally well applied to the decomposition of the Kronecker squares of the irreducible representations of the unitary groups.

III. PLETHYSMS FOR RESTRICTED GROUPS

It is well known that when a group is restricted to a subgroup, a set of functions which transforms irreducibly under the full group is, in general, reducible under the subgroup. Thus, the S functions $\{\lambda\}$ corresponding to the irreducible representation $\{\lambda\}$ of $GL(m)$ will be reducible under restriction to the rotation of symplectic groups. This means that the transformation matrix $A^{\{\lambda\}}$ for the representation $\{\lambda\}$ can be expressed as a direct sum of matrices irreducible under the lower group, e.g.,

$$A^{\{\lambda\}}=\dot{\sum}A^{[\alpha]}$$

for the rotation group, and

$$A^{\{\lambda\}}=\dot{\sum}A^{\langle\alpha\rangle}$$

for the symplectic group.

Following Littlewood,[7] we denote $[\alpha]$ and $\langle\alpha\rangle$ as the spurs of the irreducible matrices $A^{[\alpha]}$ and $A^{[\alpha]}$, respectively, i.e., $[\alpha]$ is an S function for the rotation group and $\langle\alpha\rangle$ for the symplectic group. Rules for performing the decompositions $\{\lambda\}\to\sum[\alpha]$ and $\{\lambda\}\to\sum[\alpha]$ have been given by Littlewood.[7] In many cases, the decomposition of a representation $\{\lambda\}$ of $GL(n)$ or $U(n)$ gives rise to nonstandard symbols containing more parts than allowed for the representations of the appropriate subgroup. Methods of expressing these nonstandard symbols in terms of standard symbols have been discussed by several authors.[6,7,15–17]

Plethysms for the rotation group $R(n)$ may be obtained from those established for the unitary group $U(n)$ by noting that

$$[\lambda]=\{\lambda\}+\sum(-1)^{p/2}\Gamma_{\nu\mu\lambda}\{\mu\}, \qquad (26)$$

where $\Gamma_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$ and the sum is taken over all partitions of ρ for which the Frobenius notation is[6]

$$\begin{pmatrix}r+1\\ r\end{pmatrix},\quad \begin{pmatrix}r+1, s+1\\ r\ ,\ s\end{pmatrix},\quad \begin{pmatrix}r+1, s+1, t+1\\ r\ ,\ s\ ,\ t\end{pmatrix}\cdots.$$

These partitions appear in the expansion

$$1+\sum(-1)^{p/2}\{\nu\}=1-\{2\}+\{31\}-\{41^2\}-\{3^2\}+\cdots.$$

The plethysm is then given in terms of plethysms for $U(n)$ by the expression

$$[\lambda]\otimes\{\eta\}=[\{\lambda\}+\sum(-1)^{p/2}\Gamma_{\nu\mu\lambda}\{\mu\}]\otimes\{\eta\}. \qquad (27)$$

[15] F. D. Murnaghan, *The Theory of Group Representations* (The Johns Hopkins Press, Baltimore, 1938).

[16] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952).

[17] J. D. Darling and R. G. Seyler, Acta Phys. Acad. Sci. Hung. **21**, 33 (1966).

TABLE II. Decomposition of the Kronecker squares of irreducible representations of the general rotation group.

$[\lambda]$	$[\lambda]\otimes\{2\}$	$[\lambda]\otimes\{1^2\}$
[1]	$\{2\}$	$\{1^2\}$
[2]	$\{4\}+\{22\}-\{2\}$	$\{31\}+\{0\}-\{2\}$
[3]	$\{6\}+\{42\}+\{1^2\}-\{4\}-\{31\}$	$\{51\}+\{3^2\}+\{2\}-\{4\}-\{31\}$
[4]	$\{8\}+\{62\}+\{44\}+\{31\}-\{6\}-\{51\}-\{42\}$	$\{71\}+\{53\}+\{4\}+\{2^2\}-\{6\}-\{51\}-\{42\}$
[21]	$\{42\}+\{321\}+\{31^3\}+\{2^3\}+\{1^2\}-\{31\}-\{22\}-\{211\}$	$\{41^2\}+\{3^2\}+\{321\}+\{2^21^2\}+\{2\}-\{31\}-\{2^2\}-\{21^2\}$
$[2^2]$	$\{44\}+\{42^2\}+\{3^21^2\}+\{31\}+\{2^4\}-\{42\}-\{321\}-\{2^3\}$	$\{431\}+\{4\}+\{32^21\}+\{2^2\}-\{42\}-\{321\}-\{2^3\}$
$[21^2]$	$\{42^2\}+\{41^4\}+\{3^21^2\}+\{321^3\}+\{2^4\}+\{2^21^4\}+\{21^2\}$ $-\{321\}-\{31^3\}-\{2^3\}-\{2^21^2\}-\{21^4\}$	$\{421^2\}+\{3^22\}+\{321^3\}+\{31^5\}+\{2^31^2\}+\{2^2\}+\{1^4\}$ $-\{321\}-\{31^3\}-\{2^3\}-\{2^21^2\}-\{21^4\}$
$[2^21]$	$\{4^22\}+\{4321\}+\{431^2\}+\{42^3\}+\{41^2\}+\{3^31\}+\{3^221^2\}$ $+\{3^2\}+\{32^31\}+\{32^21^3\}+\{321\}+\{2^5\}+\{2^21^2\}-\{431\}$ $-\{42^2\}-\{421^2\}-\{3^22\}-\{32^21\}-\{2^4\}-\{2^31^2\}$	$\{4^21^2\}+\{43^2\}+\{4321\}+\{42^21^2\}+\{42\}+\{3^22^2\}+\{3^221^2\}$ $+\{3^21^4\}+\{32^31\}+\{321\}+\{31^3\}+\{2^41^2\}+\{2^3\}-\{431\}$ $-\{42^2\}-\{421^2\}-\{3^22\}-\{32^21\}-\{2^4\}-\{2^31^2\}$
$[2^3]$	$\{4^3\}+\{4^22^2\}+\{43^21^2\}+\{431\}+\{42^4\}+\{3^22^21^2\}$ $+\{32^21\}-\{4^22\}-\{4321\}-\{3^221^2\}-\{32^31\}-\{2^5\}$	$\{4^231\}+\{4^2\}+\{432^21\}+\{42^2\}+\{3^31^3\}+\{3^21^2\}+\{32^41\}$ $+\{2^4\}-\{4^22\}-\{4321\}-\{3^221^2\}-\{32^31\}-\{2^5\}$

The plethysms for $\{\eta\}\equiv\{2\}$ or $\{1^2\}$ may then be readily calculated, using the results of Table I together with the special forms of Eqs. (7) and (8),

$$(A-B)\otimes\{2\}=A\otimes\{2\}+B\otimes\{1^2\}-AB$$

and

$$(A-B)\otimes\{1^2\}=A\otimes\{1^2\}+B\otimes\{2\}-AB,$$

and their extensions. For example,

$$\begin{aligned}[21]\otimes\{2\}&=[\{21\}-\{1\}]\otimes\{2\}\\&=\{21\}\otimes\{2\}+\{1\}\otimes\{1^2\}-\{21\}\{1\}\\&=\{42\}+\{31^3\}+\{321\}+\{2^3\}+\{1^2\}\\&\qquad-\{31\}-\{2^2\}-\{21^2\}.\end{aligned}$$

The decomposition of the Kronecker squares of irreducible representations of the rotation group into their symmetric and antisymmetric parts is given in Table II in terms of representations of the corresponding unitary groups. To obtain these decompositions for a particular rotation group $R(n)$, we first strike out all representations of $U(n)$ that contain more than n parts, and then decompose the remaining representations into those of $R(n)$, as in Table III.

Plethysms for the rotation and symplectic groups can also be found using the rather remarkable theorem due to Littlewood,[18] which states that if

$$\sum\Gamma_{\xi\zeta\lambda}\Gamma_{\xi\nu\mu}\{\zeta\}\{\nu\}=\sum K_{\lambda\mu\rho}\{\rho\},\tag{28a}$$

the summation on the left being with respect to all possible S functions including $\{\xi\}=\{0\}$, then

$$[\lambda][\mu]=\sum K_{\lambda\mu\rho}[\rho]\tag{28b}$$

and

$$\langle\lambda\rangle\langle\mu\rangle=\sum K_{\lambda\mu\rho}\langle\rho\rangle.\tag{28c}$$

Two special cases[18] of direct relevance to the present problem may be derived from the above result. If (μ) is a partition of 2, then:

(1) $$[\lambda]\otimes\{\mu\}=\sum H_{\lambda\mu\nu}[\nu],\tag{29a}$$

where

$$\sum H_{\lambda\mu\nu}\{\nu\}=\sum(\Gamma_{\xi\eta\lambda}\{\eta\})\otimes\{\mu\}+\sum\Gamma_{\xi\eta\lambda}\Gamma_{\xi\zeta\lambda}\{\eta\}\{\zeta\},\quad(\eta)\neq(\zeta)\tag{29b}$$

summed for all suitable S functions $\{\xi\}$, $\{\eta\}$, $\{\zeta\}$, the last term not being repeated for the interchange of $\{\eta\}$ and $\{\zeta\}$;

(2) $$\langle\lambda\rangle\otimes\{\mu\}=\sum J_{\lambda\mu\nu}\langle\nu\rangle,\tag{30a}$$

where

$$\sum J_{\lambda\mu\nu}\{\nu\}=\sum(\Gamma_{\xi\eta\lambda}\{\eta\})\otimes(\{\mu\}\cdot\{\epsilon\})+\sum\Gamma_{\xi\eta\lambda}\Gamma_{\xi\zeta\lambda}\{\eta\}\{\zeta\},\quad(\eta)\neq(\zeta),\tag{30b}$$

in which $(\epsilon)=(2)$ if $\{\xi\}$ is of *even* weight, but $(\epsilon)=(1^2)$ if $\{\xi\}$ is of *odd* weight. $(\{\mu\}\cdot\{\epsilon\})$ denotes an inner product of S functions.

The above two results do have the advantage of yielding the decomposition of the Kronecker square directly in terms of the representations appropriate to the rotation or symplectic group concerned, rather than in terms of representations of the general linear group (which must then be reduced.) It is somewhat surprising that Eqs. (28a)–(28c) have not been applied to the general problem of decomposing Kronecker products of representations of the rotation and symplectic groups more frequently. Examples of plethysms for the symplectic groups are given in Tables IV and V.

IV. KRONECKER SQUARES FOR THE GROUP G_2

Judd and Wadzinski[3] have considered the resolution of the irreducible representations contained in the decomposition of the Kronecker squares of the representations of G_2 into their symmetric and antisymmetric results. The results they give for the

[18] D. E. Littlewood, Can. J. Math. **10**, 17 (1958).

TABLE III. Decomposition of the Kronecker squares of irreducible representations of the rotation group R_5.

$[\lambda]$	$[\lambda]\otimes\{2\}$	$[\lambda]\otimes\{1^2\}$
[0]	[0]	
[1]	[2] + [0]	[11]
$[1^2]$	[22] + [2] + [1] + [0]	[21] + [11]
[2]	[4] + [22] + [2] + [0]	[31] + [11]
[21]	[42] + [40] + [32] + [31] + [30] + 2[22] + [21] + 2[2] + [11] + [1] + [0]	[41] + [33] + [32] + 2[31] + 2[21] + 2[11]
[22]	[44] + [42] + [4] + [32] + [3] + [22] + [2]	[43] + [41] + [33] + [31] + [21]

representations $(\omega 0)$ of G_2 may be obtained readily by noting that the irreducible representations of R_7 that contain only one part are irreducible under restriction to G_2; hence

$$(\omega 0)\otimes\{\lambda\} = [\omega 00]\otimes\{\lambda\} = (\{\omega\} - \{\omega - 2\})\otimes\{\lambda\}.$$

The remaining entries in their Table I may be found directly from the plethysms found for the group R_7 by noting that if Γ^a labels the representations of a group G, γ^α those of a subgroup g, and $\Gamma^a \to \sum_\alpha \gamma_a^\alpha$ under restriction to g, and, if

$$\Gamma^a \otimes \{\lambda\} = \sum_b \Gamma^b \to \sum_{b,\beta} \gamma_b^\beta, \tag{31a}$$

where $\{\lambda\} \equiv \{2\}$ or $\{1^2\}$, then

$$\left[\sum_\alpha \gamma_a^\alpha\right]\otimes\{\lambda\} = \sum_\alpha (\gamma_a^\alpha \otimes \{\lambda\}) + \sum_{\alpha<\rho} \gamma_a^\alpha \gamma_a^\rho = \sum_{b,\beta} \gamma_b^\beta. \tag{31b}$$

For example, to evaluate $(11)\otimes\{1^2\}$, we use Eq. (31a) to give

$$[110]\otimes\{1^2\} = [110] + [211] \to 2(10) + 2(11) + (20) + (21) + (30),$$

and then Eq. (31b) to yield

$$[(10) + (11)]\otimes\{1^2\} = (10)\otimes\{1^2\} + (11)\otimes\{1^2\} + (10)\times(11).$$

Using the fact that $(10)\otimes\{1^2\} = (10) + (11)$ and $(10)\times(11) = (10) + (20) + (21)$, we deduce that $(11)\otimes\{1^2\} = (11) + (30)$. The other entries of Judd and Wadzinski's Table I follow in a similar manner.

V. KRONECKER SQUARES FOR FINITE GROUPS

The separation of the Kronecker squares of the irreducible representations of a finite group G into their symmetric and antisymmetric parts has been discussed by Hamermesh[1] and Griffith.[2] The Kronecker squares of the irredicible representations $D^{(J)}$ of the three-dimensional rotation group R_3 may be readily

TABLE IV. Decomposition of the Kronecker squares of irreducible representations of the general symplectic group.

$\langle\sigma\rangle$	$\langle\sigma\rangle\otimes\{2\}$	$\langle\sigma\rangle\otimes\{1^2\}$
$\langle 0\rangle$	$\{0\}$	
$\langle 1\rangle$	$\{2\}$	$\{1^2\}$
$\langle 1^2\rangle$	$\{2^2\}+\{1^4\}-\{1^2\}$	$\{21^2\}+\{0\}-\{1^2\}$
$\langle 1^3\rangle$	$\{2^3\}+\{21^4\}+\{1^7\}-\{21^2\}-\{1^4\}$	$\{2^21^2\}+\{2\}+\{1^6\}-\{21^2\}-\{1^4\}$
$\langle 1^4\rangle$	$\{2^4\}+\{2^21^4\}+\{21^2\}+\{1^8\}-\{2^21^2\}-\{21^4\}-\{1^6\}$	$\{2^31^2\}+\{2^2\}+\{21^6\}+\{1^4\}-\{2^21^2\}-\{21^4\}-\{1^6\}$
$\langle 21\rangle$	$\{42\}+\{321\}+\{31^3\}+\{2^3\}+\{1^2\}-\{31\}-\{2^2\}-\{21^2\}$	$\{41^2\}+\{3^2\}+\{321\}+\{2^21^2\}+\{2\}-\{31\}-\{2^2\}-\{21^3\}$
$\langle 2^2\rangle$	$\{4^2\}+\{42^2\}+\{3^21^2\}+\{2^4\}+\{21^2\}-\{3^2\}-\{321\}-\{2^21^2\}$	$\{431\}+\{32^21\}+\{2^2\}+\{1^4\}-\{3^2\}-\{321\}-\{2^21^2\}$
$\langle 21^2\rangle$	$\{42^2\}+\{41^4\}+\{3^21^2\}+\{321^3\}+2\{31\}+\{2^4\}+\{2^21^4\}+3\{21^2\}+\{0\}-\{41^2\}-2\{321\}-2\{31^3\}-\{2^3\}-2\{2^21^2\}-\{21^4\}-\{2\}-\{1^2\}$	$\{421^2\}+\{4\}+\{3^22\}+\{321^3\}+\{31^5\}+\{31\}+\{2^31^2\}+2\{22\}+2\{21^2\}+\{1^4\}-2\{321\}-2\{31^3\}-\{2^3\}-\{2^21^2\}-\{21^4\}-\{2\}-\{1^2\}$
$\langle 2^21\rangle$	$\{4^22\}+\{4321\}+\{431^2\}+\{42^3\}+\{41^2\}+\{3^31\}+\{3^221^2\}+\{3^2\}+\{32^31\}+\{32^21^3\}+3\{321\}+\{31^3\}+\{2^5\}+\{2^3\}+3\{2^21^2\}+\{21^4\}+\{2\}+\{1^6\}-\{431\}-\{42^2\}-\{421^2\}-2\{3^22\}-\{3^21^2\}-3\{32^21\}-\{321^3\}-\{31\}-\{2^4\}-2\{2^31^2\}-\{2^21^4\}-\{21^2\}-\{1^4\}$	$\{4^21^2\}+\{43^2\}+\{4321\}+\{42^21^2\}+\{42\}+\{3^22^2\}+\{3^221^2\}+\{3^21^4\}+\{32^31\}+3\{321\}+2\{31^3\}+\{2^41^2\}+3\{2^3\}+2\{2^21^2\}+\{21^4\}+\{1^2\}-\{431\}-\{42^2\}-\{421^2\}-2\{3^22\}-\{3^21^2\}-3\{32^21\}-\{321^3\}-\{31\}-\{2^4\}-2\{2^31^2\}-\{2^21^4\}-\{2^2\}-2\{21^2\}-\{1^4\}$
$\langle 2^3\rangle$	$\{4^3\}+\{4^22^2\}+\{43^21^2\}+\{42^4\}+\{421^2\}+\{3^22^21^2\}+2\{3^22\}+\{32^21\}+\{321^3\}+\{31^5\}+\{2^6\}+2\{2^31^2\}+\{2^3\}+\{2^2\}+\{1^4\}+\{1^2\}+\{0\}-\{432\}-\{431^2\}-\{42^21\}-\{421^3\}-\{3^3\}-\{3^221\}-\{32^21^2\}-\{321\}-\{31^3\}-\{2^41\}-\{2^31^3\}-\{2^3\}-\{2^21^2\}-\{21^4\}-\{21^2\}$	$\{4^231\}+\{432^21\}+\{42^2\}+\{41^4\}+\{3^31^3\}+\{3^32\}+\{3^21^2\}+\{32^41\}+\{32^21\}+\{321^3\}+\{2^4\}+\{2^31^2\}+\{2^3\}+\{2^21^4\}+\{1^2\}-\{432\}-\{431^2\}-\{42^21\}-\{421^3\}-\{3^3\}-\{3^221\}-\{32^21^2\}-\{321\}-\{31^3\}-\{2^41\}-\{2^31^3\}-\{2^3\}-\{2^21^2\}-\{21^4\}$

TABLE V. Decomposition of the Kronecker squares of the irreducible representations of the symplectic group Sp_8.

$\langle\sigma\rangle$	$\langle\sigma\rangle \otimes \{2\}$	$\langle\sigma\rangle \otimes \{1^2\}$
$\langle 0\rangle$	$\langle 0\rangle$	
$\langle 1\rangle$	$\langle 2\rangle$	$\langle 1^2\rangle + \langle 0\rangle$
$\langle 1^2\rangle$	$\langle 2^2\rangle + \langle 1^4\rangle + \langle 1^2\rangle + \langle 0\rangle$	$\langle 21^2\rangle + \langle 2\rangle$
$\langle 1^3\rangle$	$\langle 2^3\rangle + \langle 21^2\rangle + \langle 2\rangle$	$\langle 2^21^2\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle$
$\langle 1^4\rangle$	$\langle 2^4\rangle + \langle 2^2\rangle + \langle 0\rangle$	$\langle 2^3\rangle + \langle 2\rangle$

decomposed into their symmetric and antisymmetric parts by first noting that for integral J

$$[D^{(J)}] = D^{(J)} \otimes \{2\} = 0, 2, \cdots, 2J$$

and

$$\{D^{(J)}\} = D^{(J)} \otimes \{1^2\} = 1, 3, \cdots, 2J - 1,$$

while for half integer J,

$$[D^{(J)}] = D^{(J)} \otimes \{2\} = 1, 3, \cdots, 2J,$$
$$\{D^{(J)}\} = D^{(J)} \otimes \{1^2\} = 0, 2, \cdots, 2J - 1,$$

and then decomposing the irreducible representations $D^{(J)}$ into those of the finite group G, followed by use of Eqs. (31a) and (31b).

For example, in the case of the icosahedral group K, we have $D^{(1)} \to T_1$.

$$D^{(1)} \otimes \{2\} = D^{(0)} + D^{(2)} \to A + V,$$

and

$$D^{(1)} \otimes \{1^2\} = D^{(1)} \to T_1,$$

giving

$$T_1 \otimes \{2\} = A + V \quad \text{and} \quad T_1 \otimes \{1^2\} = T_1.$$

The other Kronecker squares for the icosahedral group may be evaluated in a similar manner to give the results of Table VI. Similar results may be readily obtained for any other finite group that is a subgroup of R_3.

VI. SELECTION RULES AND PLETHYSM

Judd and Wadzinski[3] have considered the application of the resolution of the Kronecker squares of irreducible representations into their symmetric and antisymmetric parts to the determination of selection rules. Let us consider an operator $f^{(\Gamma_2\gamma_2)}$ that transforms as the Γ_2 representation of a group G and as the γ_2 representation of a subgroup g of G. The matrix element

$$\langle \psi^{(\Gamma_1\gamma_1)}, f^{(\Gamma_2\gamma_2)}\phi^{(\Gamma_2\gamma_2)}\rangle$$

will certainly vanish, unless either

$$\Gamma_2 \otimes \{2\} \supset \Gamma_1 \quad \text{and} \quad \gamma_2 \otimes \{2\} \supset \gamma_1$$

TABLE VI. Decomposition of the Kronecker squares of the irreducible representations of the icosahedral group K.

Γ	$\Gamma \otimes \{2\}$	$\Gamma \otimes \{1^2\}$
A	A	
T_1	$A + V$	T_1
V	$A + 2V + U$	$T_1 + T_2 + U$
T_2	$A + V$	T_2
U	$A + V + U$	$T_1 + T_2$
E'	T_1	A
E''	T_2	A
U'	$T_1 + T_2 + U$	$A + V$
W'	$2T_1 + 2T_2 + U + V$	$A + 2V + U$

or

$$\Gamma_2 \otimes \{1^2\} \supset \Gamma_1 \quad \text{and} \quad \gamma_2 \otimes \{1^2\} \supset \gamma_1.$$

These dual conditions make use of the fact that functions that transform as Γ_1 under G must be of the same symmetry classification as those spanning the γ_1 representation of the subgroup g.

As an example, consider the isoscalar factor

$$\langle [22]L \mid [20]k + [20]k\rangle,$$

where [22] and [20] are irreducible representations of R_5 and L and k are the irreducible representations $D^{[L]}$ and $D^{[k]}$ of R_3. We have

$$[20] \otimes \{2\} \supset [22] \quad \text{and} \quad [k] \otimes \{2\} \supset 0, 2, \cdots, 2k.$$

Hence, we conclude that the isoscalar factor must vanish for *odd* values of L.

Similar applications to finite groups are possible, some of which have been discussed by Hamermesh[1] and Griffith.[2]

VII. CONCLUSION

In the preceding, we have attempted to indicate some of the applications of Littlewood's technique of plethysm to the resolution of the Kronecker squares of irreducible representations into their symmetric and antisymmetric parts. Plethysm plays an important part in physics wherever the theory of groups enters. Two typical problems, where plethysm introduces remarkable simplifications, are the group theoretical analysis of the n-particle operators that may be constructed from a basic set of single-particle operators and the general problem of the classification of states of n-particle systems. Examples of these two applications will be considered in a later paper.